Chapter 1. Differentiation

Examples 1, 2 and 3 that follow revise the application of the rule:

$$\textbf{If} \quad y = ax^n \quad \textbf{then} \quad \frac{dy}{dx} = anx^{n-1}.$$

The rule does not just apply to n taking non negative integer values but is also true for n taking fractional and negative values as well.

Also remember that if $\quad y = f(x) \pm g(x)$

$$\frac{dy}{dx} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$ (The sum and difference rules.)

Example 1

Determine the gradient function, $\frac{dy}{dx}$, for each of the following.

(a) $y = 7x^5$ (b) $y = 3x^2 + 2x - 5$ (c) $y = \frac{3}{x^2}$

(d) $y = 5\sqrt{x}$ (e) $y = (x^2 + 1)(2x - 3)$

(a) If $y = 7x^5$

$$\frac{dy}{dx} = 7(5)x^{5-1} = 35x^4$$

(b) If $y = 3x^2 + 2x - 5 \quad (= 3x^2 + 2x^1 - 5x^0)$

$$\frac{dy}{dx} = 6x + 2$$

(c) If $y = \frac{3}{x^2} = 3x^{-2}$

$$\frac{dy}{dx} = -6x^{-3} = -\frac{6}{x^3}$$

(d) If $y = 5\sqrt{x} = 5x^{\frac{1}{2}}$

$$\frac{dy}{dx} = 5\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \frac{5}{2\sqrt{x}}$$

(e) If $y = (x^2 + 1)(2x - 3)$ expanding gives $y = 2x^3 - 3x^2 + 2x - 3$

Hence $\frac{dy}{dx} = 6x^2 - 6x + 2$

These same answers can be obtained from some calculators.

$\frac{d}{dx}(7x^5)$	
	$35 \cdot x^4$
$\frac{d}{dx}(3x^2+2x-5)$	
	$6 \cdot x + 2$
$\frac{d}{dx}\left(\frac{3}{x^2}\right)$	
	$\frac{-6}{x^3}$
$\frac{d}{dx}(5\sqrt{x})$	
	$\frac{5}{2 \cdot \sqrt{x}}$
$\frac{d}{dx}((x^2+1)(2x-3))$	
	$6 \cdot x^2 - 6 \cdot x + 2$

Example 2

Determine the gradient of the curve $y = x^2 + 3\sqrt{x}$ at the point (4, 22).

If $y = x^2 + 3\sqrt{x}$

$= x^2 + 3x^{\frac{1}{2}}$

then $\frac{dy}{dx} = 2x + \frac{3}{2\sqrt{x}}$.

At the point (4, 22), $x = 4$

and so $\frac{dy}{dx} = 2(4) + \frac{3}{2\sqrt{4}}$.

$= 8{\cdot}75$

$\frac{d}{dx}\left(x^2+3\sqrt{x}\right)\Big\vert x=4$	$\frac{35}{4}$

The gradient of the curve $y = x^2 + 3\sqrt{x}$ at the point (4, 22) is 8·75.

Example 3

Determine the equation of the tangent to the curve $y = \frac{4}{x}$ at the point (4, 1).

If $y = \frac{4}{x}$ i.e. $y = 4x^{-1}$

then $\frac{dy}{dx} = -4x^{-2}$

Thus at the point (4, 1), $\frac{dy}{dx} = -0{\cdot}25$.

At (4, 1) the tangent will have a gradient of −0·25.
Thus the tangent will have an equation of the form

$y = -0{\cdot}25x + c$.

$x = 4, y = 1$ must "fit": $1 = -0{\cdot}25(4) + c$

$\therefore$ $c = 2$

The required equation is $y = -0{\cdot}25x + 2$.

$\text{tangentLine}\left(\frac{4}{x}, x=4\right)$	$2 - \frac{x}{4}$

Second (and higher order) derivatives.

If $y = 2x^5$ then the gradient function, $\frac{dy}{dx}$, equals $10x^4$.

Differentiating again gives "the gradient function of the gradient function".

We call this the *second derivative* of y with respect to x and write it as $\frac{d^2y}{dx^2}$.

Thus with $y = 2x^5$, $\frac{dy}{dx} = 10x^4$ and $\frac{d^2y}{dx^2} = 40x^3$

Continuing this process: $\frac{d^3y}{dx^3} = 120x^2$, $\frac{d^4y}{dx^4} = 240x$ etc.

Alternatively, using the dash notation,
with $f(x) = 2x^5$, $f'(x) = 10x^4$, $f''(x) = 40x^3$, $f'''(x) = 120x^2$, etc.

Example 4

Find the coordinates of any points on the curve $y = 2x^3$ where the second derivative has a value of 24.

Either algebraically:

If $y = 2x^3$ then $\frac{dy}{dx} = 6x^2$

and $\frac{d^2y}{dx^2} = 12x$

Thus we require points for which $12x = 24$
i.e. $x = 2$

Or, by calculator:

solve$\left(\frac{d^2}{dx^2}(2\cdot x^3)=24, x\right)$
$\{x = 2\}$

If $x = 2$, $y = 2(2)^3$
$= 16$

Thus $y = 2x^3$ has a second derivative of 24 at (2, 16).

Exercise 1A

(Whilst you are encouraged to explore the ability of your calculator to determine expressions for the derivative, to determine its value at particular points on a curve and to find the equation of tangents to curves, it is suggested that you do most of the following questions algebraically to ensure that you can follow the basic processes without a calculator.)

Determine the gradient function $\frac{dy}{dx}$ for each of the following.

1. $y = 5x + 17$
2. $y = 3x^2 - 2x$
3. $y = 2x^3 - x^2$
4. $y = 15 - 2x$
5. $y = \frac{x}{5}$
6. $y = \frac{5}{x}$
7. $y = 3x^2 - \frac{3}{x^2}$
8. $y = 10\sqrt{x}$
9. $y = 10 + 4\sqrt{x}$
10. $y = \frac{8}{\sqrt{x}}$
11. $y = \sqrt[3]{x}$
12. $y = \frac{5x^2 - 8x}{x}$
13. $y = 6 + \frac{1}{x}$
14. $y = 5(7x^2 - 2)$
15. $y = (x^2 - 1)(3x + 2)$

Determine $\frac{d^2y}{dx^2}$ for each of the following.

16. $y = x^2$
17. $y = x^3$
18. $y = 3x^2 + x$
19. $y = 2x^3 + 2x - 34$
20. $y = 2x^2 - x - 3$
21. $y = 4x^3 + 3x^2 + 2x$
22. $y = \sqrt{x}$
23. $y = 8\sqrt{x}$
24. $y = \frac{1}{x}$
25. $y = \frac{x}{5} + 7$
26. $y = \frac{5}{x} + 7$
27. $y = x^2 + \frac{4}{x^2}$

Determine $f'(x)$ for each of the following.

28. $f(x) = 3x - \frac{1}{x}$
29. $f(x) = 5x^2 + 8\sqrt{x}$
30. $f(x) = \frac{4x^2}{\sqrt{x}}$

Determine $f''(x)$ for each of the following.

31. $f(x) = 3x^4 + 4x^3$
32. $f(x) = \frac{3}{2x^3}$
33. $f(x) = 5x^3 - \frac{1}{x^2}$

34. Find the gradient of $y = 2x^3 - 2x + 1$ at the point (1, 1).

35. Find the gradient of $y = 8 - \frac{5}{x}$ at the point (–1, 13).

36. Find the gradient of $y = 3x^2 - \frac{1}{x^2}$ at the point (–1, 2).

37. Find the value of f"(–3) for $f(x) = 2x^3 - 3x^2 + 4x + 2$.

38. If $f(x) = 5x - 2x^3$ find (a) $f'(x)$, (b) f'(2), (c) $f''(x)$, (d) f"(–2).

39. Find the equation of the tangent to the curve $y = 5x^2$ at the point (–2, 20).

40. Find the equation of the tangent to the curve $y = x + \frac{6}{x}$ at the point (2, 5).

41. Find the equation of the tangent to the curve $y = \frac{x^3 + 2\sqrt{x}}{x}$ at the point (1, 3).

42. Find the coordinates of the point(s) on the following curves where the derivative is as stated.

    (a) $y = 2x^3 + 6x^2 - 8x + 4$. $\frac{dy}{dx} = 10$.

    (b) $y = 5 + 6\sqrt{x}$. $\frac{dy}{dx} = 5$.

43. Find the coordinates of the point(s) on the following curves where the second derivative is as stated.

    (a) $y = \frac{x^3}{12}$. $\frac{d^2y}{dx^2} = 1{\cdot}5$. (b) $y = x^3 - 2x^2$. $\frac{d^2y}{dx^2} = 2$.

44. The curve $y = ax^3 + bx^2 + cx + 5$ passes through the point P (–1, 4) and at the point P the first and second derivatives of the curve are 8 and –24 respectively.
    Find the values of the constants a, b and c.

The product rule.

Consider the function $y = x(x + 3)$.

To determine $\frac{dy}{dx}$ we could simply expand the bracket to obtain $\quad y = x^2 + 3x$

and then differentiate to give $\quad \frac{dy}{dx} = 2x + 3$

Could we obtain this same answer without having to first expand $x(x + 3)$?

I.e. can we develop a rule for differentiating the product of two functions?

$$y = f(x) \times g(x)$$

Note • Initially this "product rule" for differentiating $f(x) \times g(x)$ may seem to be of limited use because expanding the expression, and then differentiating, is likely to be reasonably straightforward in many cases anyway. However such straightforward expansion may not always be the case and then the product rule can prove to be very useful.

Work through the following investigation and see if you can discover the rule for differentiating products.

<u>INVESTIGATION</u>

If $y = (x + 3)(x + 2)$ then expansion gives $y = x^2 + 5x + 6$

thus $\frac{dy}{dx} = 2x + 5$

Now $(x + 3) + (x + 2) = 2x + 5$!! Could we simply differentiate a product by summing the two parts? Clearly we need to investigate further before we can state a rule with any confidence. Copy and complete the table below and see if you can determine the rule for differentiating $y = f(x)\, g(x)$.

Function as a product	Expanded	$\frac{dy}{dx}$
$y = (x + 3)(x + 2)$	$y = x^2 + 5x + 6$	$2x + 5$
$y = (x + 7)(x + 2)$		
$y = (x + 5)(x - 3)$		
$y = (x + 5)(2x - 1)$		
$y = (x + 2)(2x - 4)$		
$y = (2x + 3)(x - 1)$		
$y = (3x + 1)(x - 1)$		
$y = (2x + 1)(3x + 2)$		
$y = (5x + 1)(2x + 3)$		
$y = (x^2 + 1)(x + 3)$		
$y = (x^2 + 3)(2x^2 + 1)$		

Did you discover a rule for differentiating a product? Well done if you did.
The product rule can be stated as follows:

$$\text{If } y = f(x)\,g(x) \text{ then } \frac{dy}{dx} = g(x)\,f'(x) + f(x)\,g'(x)$$

Alternatively, if we use u and v to represent the two functions f(x) and g(x):

$$\text{If } y = uv \text{ then } \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

i.e:

$$\frac{dy}{dx} = (\text{2nd function} \times \text{derivative of 1st}) + (\text{1st function} \times \text{derivative of 2nd})$$

As addition is commutative, i.e. a + b = b + a, this could alternatively be written:

$$\frac{dy}{dx} = (\text{1st function} \times \text{derivative of 2nd}) + (\text{2nd function} \times \text{derivative of 1st})$$

Example 5

Differentiate (a) $y = (5x - 1)(2x + 3)$ (b) $y = (3x - 5)(x^2 + 5x - 7)$

(a) $y = (5x - 1)(2x + 3)$ is of the form $y = uv$ where $u = 5x - 1$ and $v = 2x + 3$.

Using the product rule

$$\begin{aligned}\frac{dy}{dx} &= (2x + 3)(5) + (5x - 1)(2)\\ &= 10x + 15 + 10x - 2\\ &= 20x + 13\end{aligned}$$

(b) $y = (3x - 5)(x^2 + 5x - 7)$ is of the form $y = uv$ where $u = 3x - 5$ and $v = x^2 + 5x - 7$.

Using the product rule

$$\begin{aligned}\frac{dy}{dx} &= (x^2 + 5x - 7)(3) + (3x - 5)(2x + 5)\\ &= 3x^2 + 15x - 21 + 6x^2 + 5x - 25\\ &= 9x^2 + 20x - 46\end{aligned}$$

$$\frac{d}{dx}((5x - 1)(2x + 3))$$

$$20 \cdot x + 13$$

$$\frac{d}{dx}((3x - 5)(x^2 + 5x - 7))$$

$$9 \cdot x^2 + 20 \cdot x - 46$$

Exercise 1B

In this exercise many of the questions require you to "use the product rule". In such cases your method should clearly show your use of the rule. For questions that do not have such a requirement use your calculator if you wish.

1. By writing x^3 as $(x)(x^2)$ differentiate $y = x^3$ using the product rule.

Use the product rule to differentiate each of the following with respect to x.

2. $y = (x + 6)(x + 1)$
3. $y = (x + 7)(x - 3)$
4. $y = (3x + 1)(x + 4)$
5. $y = (x + 1)(3x + 4)$
6. $y = (2x + 3)(5x + 1)$
7. $y = (6x + 5)(2x + 3)$
8. $y = (x + 4)(x^2 + 2)$
9. $y = (x + 5)(x^2 - 3)$
10. $y = (x + 7)(x^2 + 1)$
11. $y = (x - 10)(x^2 + 8)$
12. $y = (2x - 1)(x^2 + 7x - 2)$
13. $y = (3x + 4)(x^2 - 3x + 4)$
14. $y = (2x - 3)(x^2 + 5x - 1)$
15. $y = (3x + 1)(x^2 - 7x + 1)$

Use the product rule to determine the gradient of each of the following at the given point.

16. $y = (x + 3)(x - 2)$ at $(3, 6)$.
17. $y = (3x + 1)(x - 5)$ at $(3, -20)$.
18. $y = (3x - 2)(2x + 1)$ at $(1, 3)$.
19. $y = (x - 4)(x^2 - 1)$ at $(2, -6)$.
20. Find the equation of the tangent to $y = (3x - 5)(x + 2)$ at the point $(2, 4)$.
21. Find the equation of the tangent to $y = (1 + 2x)(5x - 1)$ at the point $(1, 12)$.

First solve questions 22 and 23 without the assistance of your calculator then try the questions again using the ability of your calculator to determine derivatives and to solve equations.

22. Find the coordinates of any points on the curve $y = (2x - 1)(3x + 4)$ where the gradient is -1.

23. Find the coordinates of any points on the curve $y = (x - 3)(2x^2 - 11)$ where the gradient is 37.

24. Determine the coordinates of any points on the curve
$$y = (x - 3)(x^2 - 8)$$
where the gradient is the same as that of the straight line $y = x$.

25. (a) Use the product rule to differentiate $\sqrt{x^3} \times (2x + 1)$.

    (b) Differentiate $\sqrt{x^3} \times (2x + 1)$ by first expanding the bracket and then differentiating each term.

The quotient rule.

To differentiate $y = \frac{u}{v}$ where u and v are each functions of x, we use the **quotient rule**:

$$\text{If } y = \frac{u}{v} \text{ then } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 6

Differentiate with respect to x, (a) $y = \frac{3x-5}{5x-7}$ (b) $y = \frac{3x}{x^2+3}$.

(a) $y = \frac{3x-5}{5x-7}$ is of the form $y = \frac{u}{v}$ with $u = 3x - 5$
and $v = 5x - 7$.

Using the quotient rule
$$\frac{dy}{dx} = \frac{(5x-7)(3) - (3x-5)(5)}{(5x-7)^2}$$
$$= \frac{4}{(5x-7)^2}$$

(b) $y = \frac{3x}{x^2+3}$ is of the form $y = \frac{u}{v}$ with $u = 3x$
and $v = x^2 + 3$.

Using the quotient rule
$$\frac{dy}{dx} = \frac{(x^2+3)(3) - (3x)(2x)}{(x^2+3)^2}$$
$$= \frac{3x^2 + 9 - 6x^2}{(x^2+3)^2}$$
$$= \frac{9 - 3x^2}{(x^2+3)^2}$$

$$\frac{d}{dx}\left(\frac{3x-5}{5x-7}\right) \qquad \frac{4}{(5\cdot x-7)^2}$$

$$\frac{d}{dx}\left(\frac{3x}{x^2+3}\right) \qquad \frac{-3\cdot(x^2-3)}{(x^2+3)^2}$$

Exercise 1C

In this exercise many of the questions require you to "use the quotient rule". In such cases your method should show your use of the rule. For questions that do not have such a requirement use your calculator if you wish.

1. By writing x^2 as $\frac{x^5}{x^3}$ differentiate $y = x^2$ using the quotient rule.

2. Rather than differentiating $y = \frac{1}{x^n}$ by writing it as $y = x^{-n}$, use the quotient rule instead.

Use the quotient rule to differentiate each of the following with respect to x.

3. $\frac{2x}{x+3}$ 4. $\frac{3x}{5x-1}$ 5. $\frac{6x}{4x-3}$

6. $\frac{7x}{1-2x}$ 7. $\frac{5x+1}{2x+3}$ 8. $\frac{5x+1}{2x-3}$

9. $\frac{6x-1}{5x+2}$ 10. $\frac{3x-1}{2x-1}$ 11. $\frac{1-3x}{3x+1}$

12. $\frac{5x}{x^2+1}$ 13. $\frac{2x^2}{x^3+1}$ 14. $\frac{3x^2}{x^5+3}$

15. Clearly showing your use of the quotient rule, determine the gradient of the curve $y = \frac{3x}{x-2}$ at the point (4, 6).

16. Determine the gradient of the curve $y = \frac{4x}{x^2-1}$ at the point (3, 1·5).

17. Find the equation of the tangent to $y = \frac{3x+5}{x-3}$ at the point (5, 10).

18. Determine the coordinates of any points on the curve $y = \frac{2x-1}{5-4x}$ where the gradient is equal to 6.

19. (a) Differentiate $\frac{2x-3}{x}$ using the quotient rule.

(b) By writing $\frac{2x-3}{x}$ as $(2x-3)(x^{-1})$ differentiate $\frac{2x-3}{x}$ using the product rule and express your answer as a single fraction.

(c) Use the fact that $\frac{2x-3}{x} = \frac{2x}{x} - \frac{3}{x}$

$= 2 - \frac{3}{x}$ to differentiate $\frac{2x-3}{x}$.

The chain rule.

If we are told that $y = 3x^2 + 4$ we know that $\frac{dy}{dx}$, the gradient function, is $6x$.

However, suppose we are not given y directly in terms of x but instead are given y in terms of some other variable, say u, and given this other variable, u, in terms of x. Can we find $\frac{dy}{dx}$?

For example if $y = 4u + 3$ and $u = x^2 - 4$ can we find $\frac{dy}{dx}$?

We could substitute for u, from $u = x^2 - 4$, into

$$\begin{aligned} y &= 4u + 3 \quad \text{to give} \\ y &= 4(x^2 - 4) + 3 \\ &= 4x^2 - 16 + 3 \\ \text{i.e.} \quad y &= 4x^2 - 13 \\ \text{and so} \quad \frac{dy}{dx} &= 8x \end{aligned}$$

However it is possible to determine $\frac{dy}{dx}$ in terms of x, without having to first substitute for u, by using a rule called **the chain rule**:

> If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

If $\quad y = 4u + 3 \quad$ and $\quad u = x^2 - 4$

then $\quad \frac{dy}{du} = 4 \quad$ and $\quad \frac{du}{dx} = 2x$

Then, by the chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (4)(2x) \\ &= 8x, \qquad \text{as before.} \end{aligned}$$

Note: The chain rule can be remembered by imagining the "du"s cancelling:

$$\frac{dy}{dx} = \frac{dy}{\not{d}\not{u}}\frac{\not{d}\not{u}}{dx}.$$

However, as we were reminded in the *Preliminary section* at the start of this book, the terms $\frac{dy}{du}$ and $\frac{du}{dx}$ are <u>not</u> fractions, they are <u>limits of fractions</u>,

$$\frac{dy}{du} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \text{ and } \frac{du}{dx} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x}.$$

Whilst such "cancelling" is useful for recalling the rule it cannot really be carried out.

Example 7

Find $\frac{dy}{dx}$, in terms of x, given that $y = u^2 - 5u$ and $u = 7x - 3$.

If $y = u^2 - 5u$ and $u = 7x - 3$

then $\frac{dy}{du} = 2u - 5$ and $\frac{du}{dx} = 7$.

Using the chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= (2u - 5)\,7$

$= 7\,(14x - 11)$

Note:

- More "links" can be put into the chain as required. (See example 8.)
- In the above example y is a function of u and u is a function of x. Thus we have **a function of a function** or **a composite function** as encountered in the *Preliminary work* section.

$x \rightarrow$ [$u = 7x - 3$] $\rightarrow$ [$y = u^2 - 5u$] $\rightarrow y$

For example:

$x = 2 \rightarrow$ [$7(2) - 3 = 11$] $\rightarrow$ [$(11)^2 - 5(11)$] $\rightarrow y = 66$

Example 8

Find $\frac{dy}{dx}$, in terms of x, given that $y = 3t^2$, $t = 5p - 2$ and $p = 6x + 1$.

If $y = 3t^2$, $t = 5p - 2$ and $p = 6x + 1$

then $\frac{dy}{dt} = 6t$ $\frac{dt}{dp} = 5$ and $\frac{dp}{dx} = 6$.

Using the chain rule $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dp}\frac{dp}{dx}$

$= (6t)\,(5)\,(6)$

$= 180\,(5p - 2)$

$= 180(5(6x + 1) - 2)$

$= 540\,(10x + 1)$

➔ The chain rule proves to be most useful when finding $\frac{dy}{dx}$ for certain functions in which y is given directly in terms of x but for which we choose to introduce a third variable, thus allowing the chain rule to be employed. This technique is demonstrated in the next example. ⬅

Example 9

Differentiate (a) $y = (2x - 3)^4$ (b) $y = (3x^2 + 4)^5$

(a) To differentiate $y = (2x-3)^4$ let $u = 2x - 3$

then $y = u^4$.

Thus $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 2$.

By the chain rule
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= (4u^3)(2)\\ &= 4(2x-3)^3(2)\\ &= 8(2x-3)^3\end{aligned}$$

(b) To differentiate $y = (3x^2+4)^5$ let $u = 3x^2 + 4$

then $y = u^5$.

Thus $\frac{dy}{du} = 5u^4$ and $\frac{du}{dx} = 6x$.

By the chain rule
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= (5u^4)(6x)\\ &= 5(3x^2+4)^4(6x)\\ &= 30x(3x^2+4)^4\end{aligned}$$

$$\frac{d}{dx}((2x-3)^4)$$
$$8\cdot(2\cdot x-3)^3$$
$$\frac{d}{dx}((3x^2+4)^5)$$
$$30\cdot x\cdot(3\cdot x^2+4)^4$$

Points to note.

- Consider how long the previous example would have taken if we had to differentiate each part by first expanding the initial expressions (without the assistance of a calculator) and then differentiate each term!
- The final answers in the previous example are given in terms of the variable x, given in the question, and not in terms of the variable u which we introduced to help us differentiate.
- With practice you should be able to differentiate expressions like those of the previous example without having to write down the full process. (See the next example.)
- Considering the general case: If $y = [f(x)]^n$, then by letting $u = f(x)$ and using the chain rule, we obtain the following result.

$$\text{If } y = [f(x)]^n \text{ then } \frac{dy}{dx} = n\,[f(x)]^{n-1}\, f'(x)$$

Example 10

Differentiate (a) $y = (7 + 2x)^3$ (b) $y = (x^2 + 3x + 1)^6$ (c) $y = \dfrac{1}{x^3 + 1}$

(a) If $y = (7 + 2x)^3$

$$\frac{dy}{dx} = 3\,(7 + 2x)^2\,(2)$$

$$= 6\,(7 + 2x)^2$$

(b) If $y = (x^2 + 3x + 1)^6$

$$\frac{dy}{dx} = 6\,(x^2 + 3x + 1)^5\,(2x + 3)$$

$$= 6\,(2x + 3)\,(x^2 + 3x + 1)^5$$

(c) If $y = (x^3 + 1)^{-1}$

$$\frac{dy}{dx} = -1(x^3 + 1)^{-2}\,3x^2$$

$$= -\frac{3x^2}{(x^3 + 1)^2}$$

The reader should confirm that applying the quotient rule for part (c), instead of the chain rule, gives the same answer.

$$\frac{d}{dx}((7 + 2x)^3) \quad 6\cdot(2\cdot x + 7)^2$$

$$\frac{d}{dx}((x^2 + 3x + 1)^6) \quad 6\cdot(x^2 + 3\cdot x + 1)^5\cdot(2\cdot x + 3)$$

$$\frac{d}{dx}\left(\frac{1}{x^3 + 1}\right) \quad \frac{-3\cdot x^2}{(x^3 + 1)^2}$$

Example 11

Determine the gradient of the curve $y = (x^2 - 7)^4$ at the point (3, 16).

Either algebraically

If $y = (x^2 - 7)^4$

$$\frac{dy}{dx} = 4(x^2 - 7)^3 (2x)$$

$$= 8x(x^2 - 7)^3$$

At (3, 16), $x = 3$

and $\frac{dy}{dx} = 24(3^2 - 7)^3$

$$= 24 \times 8$$

$$= 192.$$

or by calculator

$\frac{d}{dx}((x^2-7)^4)|x=3$

192

The gradient of the curve $y = (x^2 - 7)^4$ at the point (3, 16) is 192.

Exercise 1D

1. Find $\frac{dy}{dx}$, in terms of x, given that $y = 7u - 3$ and $u = 2x^2 + 5x - 3$.

2. Find $\frac{dp}{dt}$, in terms of t, given that $p = 3s^2$ and $s = 2t + 1$.

3. Find $\frac{dh}{dr}$, in terms of r, given that $h = 5p^2 - 3$ and $p = 1 - 2r^2$.

4. Find $\frac{dy}{dx}$, in terms of x, given that $y = u^2 + 3$, $u = 4p - 3$ and $p = 3x + 2$.

5. Differentiate $y = (3x + 2)^5$ by letting $u = 3x + 2$ and using the chain rule. Show your working fully and give your answer in terms of x.

6. Differentiate $y = (x^2 + 2)^3$ by letting $u = x^2 + 2$ and using the chain rule. Show your working fully and give your answer in terms of x.

7. Differentiate $y = \frac{1}{(8x - 3)}$ by letting $u = 8x - 3$ and using the chain rule. Show your working fully and give your answer in terms of x.

8. Differentiate $y = \sqrt{2x + 3}$ by letting $u = 2x + 3$ and using the chain rule. Show your working fully and give your answer in terms of x.

9. Differentiate $y = \dfrac{1}{\sqrt{6x+1}}$ by letting $u = 6x + 1$ and using the chain rule. Show your working fully and give your answer in terms of x.

10. Differentiate $y = \dfrac{1}{(3x^2+2x+1)^2}$ by letting $u = 3x^2 + 2x + 1$ and using the chain rule. Show your working fully and give your answer in terms of x.

Find the gradient function $\dfrac{dy}{dx}$ for each of the following. Do each one without the assistance of a calculator and then check your answer with your calculator.

11. $y = (5x+2)^4$

12. $y = (7x-3)^3$

13. $y = (2-3x)^3$

14. $y = (4+7x)^2$

15. $y = (3x^2+5)^3$

16. $y = (2x^3+1)^6$

17. $y = (x+2)^{-3}$

18. $y = (2x+5)^{-1}$

19. $y = \dfrac{1}{(x+2)}$

20. $y = \dfrac{1}{(7x-3)^2}$

21. $y = 3x + (2x+3)^5$

22. $y = \sqrt{x+1}$

Determine the gradient of each of the following at the given point without the assistance of your calculator.

23. $y = (10x+1)^5$ at $(0, 1)$.

24. $y = (6x-1)^3$ at $(1, 125)$.

25. $y = (1+x^4)^3$ at $(-1, 8)$.

26. $y = (2x-3)^{-4}$ at $(2, 1)$.

27. $y = \dfrac{1}{(2x^2+1)^3}$ at $(0, 1)$.

28. $y = x^2 + (x-1)^5$ at $(2, 5)$.

Use your calculator to determine the gradient of each of the following at the given point.

29. $y = \dfrac{1}{3+2x+3x^2}$ at $(1, 0{\cdot}125)$.

30. $y = \dfrac{40}{\sqrt{1+x}}$ at $(3, 20)$.

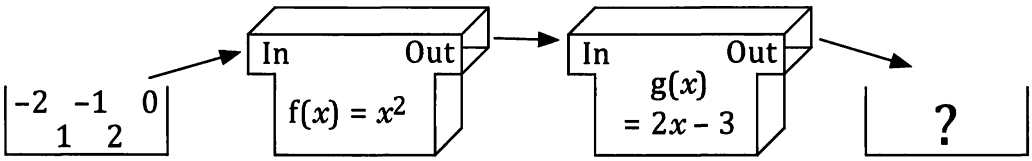
31. $y = \dfrac{36}{1+\sqrt{x}}$ at $(4, 12)$.

Miscellaneous Exercise One.

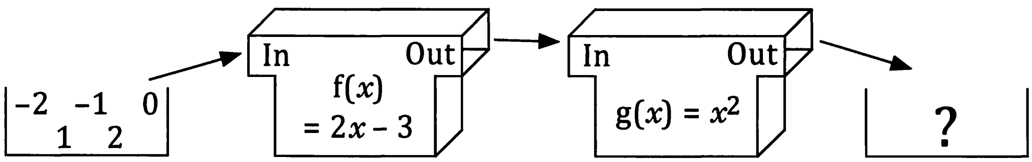
This miscellaneous exercise may include questions involving the work of this chapter and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Each of the following diagrams show a composite function $gf(x)$.
With the domain as shown determine the range of $gf(x)$.

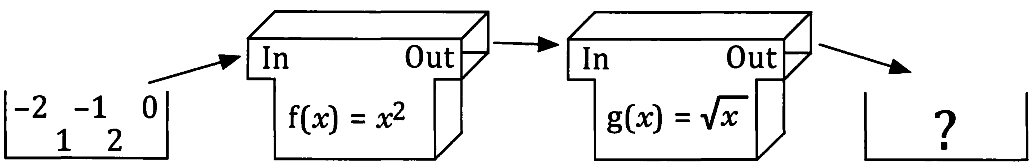
(a)



(b)



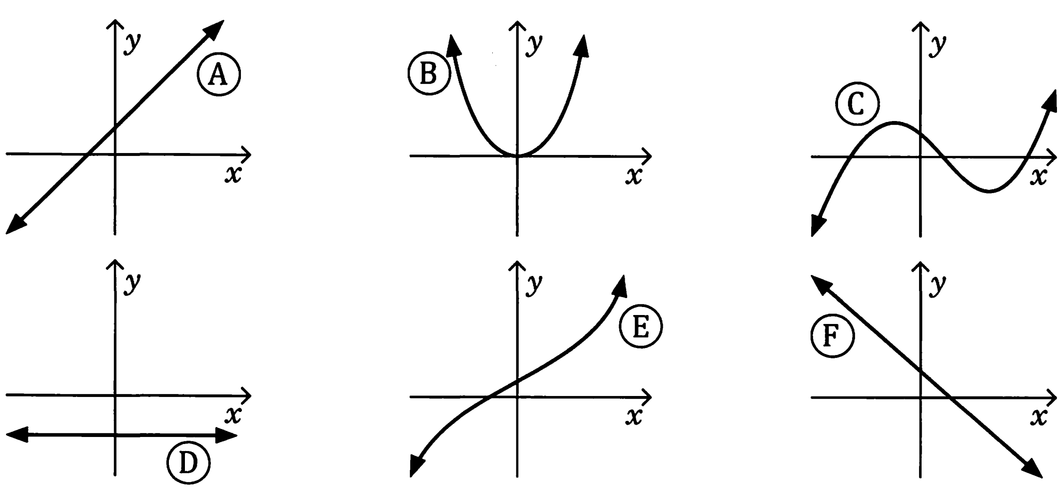
(c)



2. For the graphs A to F shown below state which have

(a) $\frac{dy}{dx}$ always positive, (b) $\frac{dy}{dx}$ always negative,

(c) $\frac{dy}{dx}$ never negative, (d) $\frac{dy}{dx}$ independent of x.



3. (You should be able to do this question mentally and simply write the answer.)

   If $y = 5 - 7x^2$ determine $\dfrac{d^2y}{dx^2}$.

4. Find $\dfrac{dy}{dx}$ for (a) $y = 5x^2$ (b) $y = 3 + 5x^2$ (c) $y = (3 + 5x)^2$.

5. Clearly showing the use of the product rule,

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx},$$

   determine $\dfrac{dy}{dx}$ for each of the following:

   (a) $y = (x + 1)(x - 3)$ (b) $y = (2x - 1)(5x + 4)$

   (c) $y = (2x + 3)^2$ (d) $y = (x^2 - 4)(3x + 5)$

6. Find the gradient of $y = 2(x^2 - 5)^7$ at the point $(-2, -2)$.

7. Differentiate $\dfrac{x^3 - 3x^2}{x}$. (Hint: You do not need the quotient rule for this one.)

8. Find the gradient of $y = \dfrac{4}{2x + 3}$ at the point $(-1, 4)$.

9. Find the equation of the tangent to $y = \dfrac{2x - 3}{x + 1}$ at $(3, 0{\cdot}75)$, giving your answer in the form $ay = bx + c$, with a, b and c taking integer values.

10. With the assistance of your calculator:
    Find the coordinates of the points where the curve

$$y = \frac{13x + 1}{2x + 2}$$

    cuts the line $y = x + 2$.
    Find the gradient of the curve at each of these points.

11. (a) Use the product rule to obtain the derivative of $(x + 4)(2x - 1)$.

    (b) Hence, and without the assistance of a calculator, determine the derivative of $(3x - 1)(x + 4)(2x - 1)$.